Homology cycles in manifolds with locally standard torus actions

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Abstract. Let X be a $2n$-manifold with a locally standard action of a compact torus T^n. If the free part of action is trivial and proper faces of the orbit space Q are acyclic, then there are three types of homology classes in X: (1) classes of face submanifolds; (2) k-dimensional classes of Q swept by actions of subtori of dimensions $< k$; (3) relative k-classes of Q modulo ∂Q swept by actions of subtori of dimensions $\geqslant k$. The submodule of $H_*(X)$ spanned by face classes is an ideal in $H_*(X)$ with respect to the intersection product. It is isomorphic to $(\mathbb{Z}[S_Q]/\Theta)/W$, where $\mathbb{Z}[S_Q]$ is the face ring of the Buchsbaum simplicial poset S_Q dual to Q; Θ is the linear system of parameters determined by the characteristic function; and W is a certain submodule, lying in the socle of $\mathbb{Z}[S_Q]/\Theta$. Intersections of homology classes different from face submanifolds are described in terms of intersections on Q and T^n.

1. Introduction

An action of a compact torus T^n on a smooth compact manifold M of dimension $2n$ is called locally standard if it is locally isomorphic to the standard action of T^n on $\mathbb{C}^n$. The orbit space $Q = M/T^n$ has a natural structure of a manifold with corners in which open k-dimensional faces of Q correspond to k-dimensional orbits of an action. Every manifold with locally standard torus action is equivariantly homeomorphic to the quotient model $X = Y/\sim$, where Y is a principal T^n-bundle over Q and $\sim$ is an equivalence relation determined by the characteristic function on Q [**17**].

This paper is the third in a series of works, where we study topology of X under the assumption that proper faces of the orbit space are acyclic and Y is a trivial

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bundle. Previous works [**1, 2**] were devoted to the homological spectral sequence associated with the filtration of X by orbit types. In this paper we give a geometrical description of homology cycles on X.

In the case when all faces of Q including Q itself are acyclic, the topology of the corresponding manifold X is known (see [**9**]). In this case the equivariant cohomology ring is isomorphic to the face ring of the simplicial poset S_Q dual to Q: $H_T^*(X;\mathbb{Z}) \cong \mathbb{Z}[S_Q]$. As a ring, it is generated by equivariant cycles, dual to face submanifolds of X (these generators correspond to the standard generators of the face ring). The spectral sequence of the Borel fibration $ET^n \times_T X \to BT^n$ collapses at a second page. A fiber-inclusion map $\iota\colon X \hookrightarrow ET^n \times_T X$ induces a surjective ring homomorphism $\iota^*\colon H_T^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z})$, whose kernel is the image of $H^{>0}(BT^n;\mathbb{Z})$ under π^*. Thus, $H^*(X;\mathbb{Z}) \cong \mathbb{Z}[S_Q]/(\theta_1,\ldots,\theta_n)$, where θ_i are the images of generators v_i of the ring $H^*(BT^n;\mathbb{Z}) \cong \mathbb{Z}[v_1,\ldots,v_n]$. The sequence $(\theta_1,\ldots,\theta_n)$ is a linear system of parameters in $\mathbb{Z}[S_Q]$. Since S_Q is Cohen–Macaulay, this is a regular sequence and $\dim H^{2k}(X) = h_k(S_Q)$.

In the case when only *proper* faces of Q are acyclic, this approach is inapplicable. The spectral sequence of the Borel fibration does not collapse at a second page. We still have the ring homomorphism $H_T^*(X)/(\pi^*H^{>0}(BT^n)) \to H^*(X)$, but it is neither injective nor surjective.

Nevertheless, there is an apparent connection between topology of spaces with torus actions and the theory of face rings. In [**3**] we proved that there exists an isomorphism of rings (and $H^*(BT^n)$-modules):

$$H_T^*(X;\mathbb{Z}) \cong \mathbb{Z}[S_Q] \oplus H^*(Q;\mathbb{Z}) \tag{1.1}$$

(the units of the rings in the direct sum are identified).

When proper faces of Q are acyclic, the dual simplicial poset S_Q is Buchsbaum [**1**, Cor.6.3]. There is a standard tool in combinatorics and commutative algebra devised to study Buchsbaum simplicial complexes, namely, the h'-vector. By definition, the h'-numbers of a Buchsbaum simplicial poset S are the dimensions of homogeneous components of the quotient algebra $\Bbbk[S]/(\theta_1,\ldots,\theta_n)$, where $\theta_1,\ldots,\theta_n$ is any linear system of parameters. These numbers do not depend on a linear system of parameters, and can be expressed in terms of the ordinary h-numbers and Betti numbers of S (see [**14, 11**] or Definition 2.8 below). In [**2**, Th.3] we proved that $\dim(E_X)^2_{q,q} = h'_{n-q}(S_Q)$, where $(E_X)^*_{*,*}$ is the homological spectral sequence associated with the orbit type filtration of X.

In this paper we describe the geometrical structure of homology cycles on X.

THEOREM 1. *Homology classes of X have three different types:*

(1) *the classes of face submanifolds (we call them face classes);*
(2) *the classes, represented by k-cycles of Q, swept by an action of a subtorus of dimension $< k$ (these classes will be called spine classes);*

(3) *the classes, represented by relative k-cycles of Q modulo ∂Q with $k < n$, swept by an action of a subtorus of dimension $\geqslant k$ (these classes will be called diaphragm classes).*

Linear relations on face classes are of two types: the relations appearing in the ring $\Bbbk[S_Q]/(\theta_1, \ldots, \theta_n)$, and additional relations lying in a socle of $\Bbbk[S_Q]/(\theta_1, \ldots, \theta_n)$.

Intersections of face classes are encoded by the multiplication in the face ring of S_Q. Proper face classes span the ideal of $H_*(X)$ with respect to the intersection product. Intersections of other classes are described by means of the intersection products on Q and T^n.

Precise statements are given in Propositions 2.9, 4.1, 5.6, 6.1, 6.2, 6.3, and Theorem 2.

Face classes and the elements of $H_k(Q, \partial Q)$ swept by the action of the whole group T^n are equivariant. This gives an independent geometrical evidence for the formula (1.1).

The paper may be briefly outlined as follows. Section 2 contains basic definitions and outlines the previous results. In Section 3 we make technical preparations for Section 4, which is devoted to linear relations on face classes. In Section 5 we realize non-face classes of X as embedded pseudomanifolds. These geometrical constructions imply a partial description of intersection theory on X, which is done in Section 6.

Two examples of computations are discussed in Section 7. A very particular 4-dimensional example is worked out, and the reader is encouraged to refer to it while reading other parts of the paper. The second example is more general: we apply our technique to the class of orientable toric origami manifolds with acyclic proper faces of the orbit space, and rediscover some results of [**3**]. A supplementary space $\widehat{X}$ is introduced in the last section. This space can be considered as a T^n-invariant tubular neighborhood of the union of characteristic submanifolds in X. By using intersection theory on $\widehat{X}$, we prove that certain elements of $\Bbbk[S_Q]/(\theta_1, \ldots, \theta_n)$ lie in the socle of this module. This gives a geometrical interpretation of the result obtained by Novik–Swartz [**11**].

2. Preliminaries and previous results

2.1. Manifolds with locally standard torus actions. An action of T^n on a (compact connected smooth) manifold M^{2n} is called *locally standard*, if M has an atlas of T^n-invariant charts, each equivalent to an open T^n-invariant subset of the standard action of T^n on $\mathbb{C}^n$. The reader is referred to [**6**] or [**17**] for the precise definition. The orbit space of a locally standard action is a compact connected n-dimensional manifold with corners with the property that every codimension k face of Q lies in exactly k facets of Q (such manifolds with corners were called *nice* in [**9**], or *manifolds with faces* elsewhere).

Definition 2.1. *A finite partially ordered set (poset) S is called simplicial if (1) there is a minimal element $\hat{0} \in S$; (2) for each element $J \in S$ the lower order ideal $\{I \in S \mid I \leqslant J\}$ is isomorphic to the poset of faces of a k-simplex for some number k, called the dimension of I.*

The elements of S are called simplices. Simplices of dimension 0 are called vertices. The number $|I| = \dim I + 1$ is equal to the number of vertices of I and is called the rank of I. The set of vertices of a simplicial poset or a simplex is denoted by $\mathrm{Vert}(\cdot)$.

Every manifold with corners Q determines a dual poset S_Q whose elements are the faces of Q ordered by the reversed inclusions. When Q is a nice connected manifold with corners, S_Q is a simplicial poset. We denote abstract elements of S_Q by I, J, etc. and the corresponding faces of Q by F_I, F_J, etc. There holds $\dim F_I = n - |I|$. The minimal element of S_Q corresponds to the maximal face of Q, i.e. Q itself. Vertices of S_Q correspond to facets of Q. The set of facets of Q is denoted by $\mathrm{Fac}(Q)$.

Let Q be the orbit space of locally standard action, and let $x \in F^\circ$ be a point in the interior of a facet $F \in \mathrm{Fac}(Q)$. Then the stabilizer of x, denoted by $\lambda(F)$, is a 1-dimensional toric subgroup in T^n. If F_I is a codimension k face of Q, contained in facets $F_1, \ldots, F_k \in \mathrm{Fac}(Q)$, then the stabilizer of $x \in F_I^\circ$ is the k-dimensional torus $T_I = \lambda(F_1) \times \ldots \times \lambda(F_k) \subset T^n$, where the product is free inside T^n. This puts a specific restriction on subgroups $\lambda(F)$. In general, a map

$$\lambda\colon \mathrm{Fac}(Q) \to \{\text{1-dimensional toric subgroups of } T^n\} \tag{2.1}$$

is called *a characteristic function*, if, whenever facets $F_1, \ldots, F_k$ have nonempty intersection, the map

$$\lambda(F_1) \times \ldots \times \lambda(F_k) \to T^n,$$

induced by inclusions $\lambda(F_i) \hookrightarrow T^n$, is injective and splits. This condition is called $(*)$-condition. Let $i \in \mathrm{Vert}(S_Q)$ be the vertex of S_Q, and $T_i = \lambda(F_i)$ be the value of characteristic function. Let $\omega_i \in H_1(T^n; \Bbbk) \cong \Bbbk^n$ be the fundamental class of T_i. This class is defined uniquely up to sign.

Let $\mu\colon M \to Q$ be the projection to the orbit space. The free part of the action has the form $\mu|_{Q^\circ}\colon \mu^{-1}(Q^\circ) \to Q^\circ$, where $Q^\circ = Q \backslash \partial Q$ is the interior of the manifold with corners. It is a principal torus bundle over Q° which can be uniquely extended over Q; it determines a principal T^n-bundle $\rho\colon Y \to Q$. Therefore any manifold with locally standard action defines three objects: a nice manifold with corners Q, a principal torus bundle $\rho\colon Y \to Q$, and a characteristic function λ. One can recover the manifold M, up to equivariant homeomorphism, from these data by the following standard construction.

Construction 2.2 (Quotient construction). Let $\rho\colon Y \to Q$ be a principal T^n-bundle over a nice manifold with corners, and λ be a characteristic function on $\mathrm{Fac}(Q)$. Consider the space $X \overset{\text{def}}{=} Y/\sim$, where $y_1 \sim y_2$ if and only if $\rho(y_1) = \rho(y_2) \in$

F_I° for some face F_I of Q, and y_1, y_2 lie in the same T_I-orbit of the T^n-action on Y. Let $f\colon Y \to X$ be the quotient map.

Every manifold M with locally standard torus action is equivariantly homeomorphic to its model X ([**17**, Cor.2]). In the rest of the paper we use the model X instead of M.

REMARK 2.3. In the paper we work with a smooth manifold with corners Q and smooth manifolds $X \cong M$, but this is done basically to simplify the exposition. The quotient model $X = (Q \times T^n)/\sim$ can obviously be defined for a larger class of spaces. If Q is a homology manifold with a simple stratification of the boundary, in which faces are homology manifolds with boundaries, then X is a closed homology manifold. All results of the paper are valid in this setting as can be seen from the proofs.

2.2. Filtrations. There are natural topological filtrations on Q, Y and X. Namely, $Q_k \subseteq Q$ is the union of k-dimensional faces of Q, $Y_k = \rho^{-1}(Q_k) \subseteq Y$, and $X_k = f(Y_k) \subset X$ is the union of toric orbits of dimension at most k. The maps $\mu\colon X \to Q$, $\rho\colon Y \to Q$ and $f\colon Y \to X$ respect these filtrations. The homological spectral sequences produced by these filtrations are denoted $(E_Q)^*_{*,*}$, $(E_Y)^*_{*,*}$, and $(E_X)^*_{*,*}$. The map f induces the morphism of spectral sequences $f^r_*\colon (E_Y)^r \to (E_X)^r$.

The subsets $\rho^{-1}(F_I) \subset Y$ and $\mu^{-1}(F_I) \subset X$ which cover the face $F_I \subset Q$ are denoted Y_I and X_I respectively. Note that the subset X_I is a closed submanifold of X of codimension $2|I|$. It is called a *face submanifold*. Face submanifolds of codimension 2 are called *characteristic submanifolds*. They correspond to facets of Q.

The first page of $(E_Q)^*_{*,*}$ has the form

$$(E_Q)^1_{p,q} = H_{p+q}(Q_p, Q_{p-1}) \cong \bigoplus_{I \in S_Q, \dim F_I = p} H_{p+q}(F_I, \partial F_I).$$

and the first differential $(d_Q)^1$ is the sum of the maps

$$\begin{aligned}(2.2)\quad m^q_{I,J}\colon H_{q+\dim F_I}(F_I, \partial F_I) \to H_{q+\dim F_I - 1}(\partial F_I) \to \\ \to H_{q+\dim F_I-1}(\partial F_I, \partial F_I \backslash F_J^\circ) \cong H_{q+\dim F_J}(F_J, \partial F_J),\end{aligned}$$

defined for every face F_I and $F_J \in \mathrm{Fac}(F_I)$. Here the first map is the connecting homomorphism in the homology exact sequence of $(F_I, \partial F_I)$, and the last isomorphism is due to excision.

2.3. Almost acyclic case. Let $\Bbbk$ be a ground ring. When coefficients in the notation of (co)homology are omitted, they are supposed to be in $\Bbbk$. From now on we impose two restrictions on X mentioned in the introduction. First, Q is an orientable manifold and all its proper faces are acyclic (over $\Bbbk$). Second, the principal torus bundle $Y \to Q$ is trivial. Thus $X = (Q \times T^n)/\sim$. The following propositions were proved in [**1**], [**2**].

PROPOSITION 2.4. *The poset S_Q is a Buchsbaum simplicial poset (over $\Bbbk$).*

PROPOSITION 2.5. *There exists a homological spectral sequence $(\dot{E}_Q)^r_{p,q} \Rightarrow H_{p+q}(Q)$, $(\dot{d}_Q)^r\colon (\dot{E}_Q)^r_{p,q} \to (\dot{E}_Q)^r_{p-r,q+r-1}$ with the properties:*

1. $(\dot{E}_Q)^1 = H((E_Q)^1, d_Q^-)$, *where the differential* $d_Q^-\colon (E_Q)^1_{p,q} \to (E_Q)^1_{p-1,q}$ *coincides with* $(d_Q)^1$ *for* $p < n$, *and vanishes otherwise.*
2. *The module* $(\dot{E}_Q)^r_{*,*}$ *coincides with* $(E_Q)^r_{*,*}$ *for* $r \geqslant 2$.
3. $$(\dot{E}_Q)^1_{p,q} = \begin{cases} H_p(\partial Q), & \text{if } q = 0, p < n; \\ H_{q+n}(Q, \partial Q), & \text{if } p = n, q \leqslant 0; \\ 0, & \text{otherwise.} \end{cases}$$
4. *Nontrivial differentials for* $r \geqslant 1$ *have pairwise different domains and targets. They have the form* $(\dot{d}_Q)^r\colon (\dot{E}_Q)^r_{n,1-r} \to (\dot{E}_Q)^r_{n-r,0}$ *and coincide with the connecting homomorphisms* $\delta_{n+1-r}\colon H_{n+1-r}(Q, \partial Q) \to H_{n-r}(\partial Q)$.

Let Λ_* denote the homology module of a torus: $\Lambda_* = \bigoplus_s \Lambda_s$, $\Lambda_s = H_s(T^n)$.

PROPOSITION 2.6. *There exists a homological spectral sequence $(\dot{E}_Y)^r_{p,q} \Rightarrow H_{p+q}(Y)$ such that*

1. $(\dot{E}_Y)^1 = H((E_Y)^1, d_Y^-)$, *where the differential* $d_Y^-\colon (E_Y)^1_{p,q} \to (E_Y)^1_{p-1,q}$ *coincides with* $(d_Q)^1$ *for* $p < n$, *and vanishes otherwise.*
2. $(\dot{E}_Y)^r = (E_Y)^r$ *for* $r \geqslant 2$.
3. $(\dot{E}_Y)^r_{p,q} = \bigoplus_{q_1+q_2=q} (\dot{E}_Q)^r_{p,q_1} \otimes \Lambda_{q_2}$ *and* $(\dot{d}_Y)^r = (\dot{d}_Q)^r \otimes \mathrm{id}_\Lambda$ *for* $r \geqslant 1$.

PROPOSITION 2.7. *There exists a homological spectral sequence $(\dot{E}_X)^r_{p,q} \Rightarrow H_{p+q}(X)$ and the morphism of spectral sequences $\dot{f}^r_*\colon (\dot{E}_Y)^r_{*,*} \to (\dot{E}_X)^r_{*,*}$ such that:*

1. $(\dot{E}_X)^1 = H((E_X)^1, d_X^-)$ *where the differential* $d_X^-\colon (E_X)^1_{p,q} \to (E_X)^1_{p-1,q}$ *coincides with* $(d_X)^1$ *for* $p < n$, *and vanishes otherwise. The map* $\dot{f}^1_*$ *is induced by* $f^1_*\colon (E_Y)^1 \to (E_X)^1$.
2. $(\dot{E}_X)^r = (E_X)^r$ *for* $r \geqslant 2$.
3. $(E_X)^1_{p,q} = (\dot{E}_X)^1_{p,q} = 0$ *for* $p < q$.
4. $\dot{f}^1_*\colon (\dot{E}_Y)^1_{p,q} \to (\dot{E}_X)^1_{p,q}$ *is an isomorphism for* $p > q$ *and injective for* $p = q$.
5. *As a consequence of previous items, for* $r \geqslant 1$, *the differentials* $(\dot{d}_X)^r$ *are either isomorphic to* $(\dot{d}_Y)^r$ *(when they hit the cells with* $p > q$*), or isomorphic to the composition of* $(\dot{d}_Y)^r$ *with* $\dot{f}^r_*$ *(when they hit the cells with* $p = q$*), or zero (otherwise).*
6. *The ranks of diagonal terms at a second page are the* h'*-numbers of the poset* S_Q *dual to the orbit space:* $\dim(\dot{E}_X)^2_{q,q} = \dim(E_X)^2_{q,q} = h'_{n-q}(S_Q)$.

Recall the definition of h'-numbers.

DEFINITION 2.8. *Let S be a pure simplicial poset,* $\dim S = n-1$. *Let f_k be the the number of k-dimensional simplices in S. The array $(f_{-1}=0, f_0, \ldots, f_{n-1})$ is called the f-vector of S. Define h-numbers by the relation:*

$$h_0 s^n + h_1 s^{n-1} + \ldots + h_n = f_{-1}(s-1)^n + f_0(s-1)^{n-1} + \ldots + f_{n-1}.$$

Let $\widetilde{\beta}_k(S) = \dim \widetilde{H}_k(S)$. Define h'-numbers by the relation

$$h'_k = h_k + \binom{n}{k}\left(\sum_{j=1}^{k-1}(-1)^{k-j-1}\widetilde{\beta}_{j-1}(S)\right) \text{ for } 0 \leqslant k \leqslant n.$$

Propositions 2.5–2.7 yield the description of $H_*(X)$. Let $H_{k,l}(Y)$ denote the $\Bbbk$-module $H_k(Q)\otimes\Lambda_l$. By Künneth's formula, $H_j(Y) \cong \bigoplus_{k+l=j} H_{k,l}(Y)$.

PROPOSITION 2.9. *Over a field, there exists a decomposition $H_j(X) \cong \bigoplus_{k+l=j} H_{k,l}(X)$ and the $\Bbbk$-module homomorphisms $f_*\colon H_{k,l}(Y) \to H_{k,l}(X)$ with the following properties:*

1. *If $k > l$, then $f_*\colon H_{k,l}(Y) \to H_{k,l}(X)$ is an isomorphism. In particular, $H_{k,l}(X) \cong H_k(Q)\otimes\Lambda_l$.*
2. *If $k < l$, there exists an isomorphism $H_{k,l}(X) \cong H_k(Q,\partial Q)\otimes\Lambda_l$.*
3. *If $k < n$, the module $H_{k,k}(X)$ fits in the exact sequence*
$$0 \to (\dot{E}_X)^{\infty}_{k,k} \to H_{k,k}(X) \to H_k(Q,\partial Q)\otimes\Lambda_k \to 0.$$
4. *$H_{n,n}(X) \cong \Bbbk$.*

There holds bigraded Poincare duality: $H_{k,l}(X) \cong H_{n-k,n-l}(X)$.

3. Preliminary computations

3.1. Orientations. We use the notation $I \overset{k}{<} J$ whenever the simplices $I, J \in S$ satisfy $I < J$ and $|J| - |I| = k$. For each pair $I \overset{2}{<} J$, there are exactly two simplices $J' \neq J''$ between them: $I \overset{1}{<} J', J'' \overset{1}{<} J$. For every simplicial poset, there exists a "sign convention" which means that we can associate an incidence number $[J\!:\!I] = \pm 1$ to any pair $I \overset{1}{<} J \in S$ in such way that the relation $[J\!:\!J']\cdot[J'\!:\!I] + [J\!:\!J'']\cdot[J''\!:\!I] = 0$ holds for any $I \overset{2}{<} J$.

The choice of a sign convention is equivalent to the choice of orientations of all nonempty simplices. By the orientation of a simplex I in an abstract simplicial poset we mean the rule which tells whether a given total ordering of the vertices of I is positive or negative, so that even permutations of the order preserve the sign and odd permutations change it. If $I \overset{1}{<} J$, then there is exactly one vertex i of J which is not in I. Given the orientations of simplices I and J, and given some positive ordering $i_1 < \ldots < i_s$ of the vertices of I, we set $[J\!:\!I]$ to be $+1$ if $i < i_1 < \ldots < i_s$ is a positive ordering on $\mathrm{Vert}(J)$, and -1 if it is negative. The construction works

in the opposite direction in an obvious way: incidence signs determine orientations of all simplices by induction.

Fix arbitrary orientations of the orbit space Q and the torus T^n. Together they define an orientation of $Y = Q\times T^n$ and $X = Y/\sim$. Also choose an *omniorientation*, which means the orientations of all characteristic submanifolds $X_{\{i\}}$. A choice of an omniorientation determines the characteristic values $\omega_i \in H_1(T^n;\mathbb{Z})$ without ambiguity of sign. To perform explicit calculations with the spectral sequences $(\dot{E}_X)^*$ and $(\dot{E}_Y)^*$ we also need to orient all faces of Q.

CONSTRUCTION 3.1. The orientation of a simplex $I \in S_Q$ determines the orientation of a face $F_I \subset Q$ by the following convention.

Let $i_1,\ldots,i_{n-q}$ be the vertices of I, listed in a positive order. The face F_I lies in the intersection of facets $F_{i_1},\ldots,F_{i_{n-q}}$. The normal bundles ν_i to facets F_i have natural orientations, in which inward normal directions are positive. Orient F_I in such way that $T_xF_I \oplus \nu_{i_1} \oplus \ldots \oplus \nu_{i_{n-q}} \cong T_xQ$ is positive in the orientation of Q.

Thus there are distinguished elements $[F_I] \in H_{\dim F_I}(F_I,\partial F_I)$. By checking the signs one can prove that the maps

$$m^0_{I,J}\colon H_{\dim F_I}(F_I,\partial F_I) \to H_{\dim F_J}(F_J,\partial F_J)$$

(see (2.2)) send $[F_I]$ to $[J:I]\cdot[F_J]$.

The choice of omniorientation and orientations of simplices determines the orientation of each orbit T^n/T_I by the following convention.

CONSTRUCTION 3.2. Let $i_1,\ldots,i_{n-q}$ be the vertices of I, listed in a positive order. The module $H_1(T^n/T_I)$ is naturally identified with Λ_1/L_I, where L_I is a submodule generated by $\omega_{i_1},\ldots,\omega_{i_{n-q}} \in \Lambda_1 = H_1(T^n)$. The basis $[\gamma_1],\ldots,[\gamma_q] \in H_1(T^n/T_I)$, $[\gamma_l] = \gamma_l + L_I$ is said to be positive if the basis $(\omega_{i_1},\ldots,\omega_{i_{n-q}},\gamma_1,\ldots,\gamma_q)$ is positive in Λ_1. This orientation of T^n/T_I determines a distinguished fundamental cycle $\Omega_I \in H_q(T^n/T_I)$.

The omniorientation and the orientation of S together determine the class of each face submanifold: $[X_I] = [F_I]\otimes\Omega_I$. Note that both orientations $[F_I]$ and $[\Omega_I]$ depend on the orientation of I by construction. Thus $[X_I]$ does not actually depend on the sign convention on S_Q and depends only on the omniorientation.

3.2. Arithmetics of torus quotients. Let us fix some coordinate representation of the torus $T^n = T^{(\{1\})}\times\ldots\times T^{(\{n\})}$, where each $T^{(\{j\})}$ is a 1-dimensional torus with a chosen orientation. For a subset $A = \{j_1 < \ldots < j_q\} \subseteq [n]$ we denote the coordinate subtorus $T^{(\{j_1\})}\times\ldots\times T^{(\{j_q\})} \subseteq T^n$ by $T^{(A)}$.

The coordinate splitting gives a positive basis $e_1,\ldots,e_n$ of the module $\Lambda_1 = H_1(T^n)$, where e_j is the fundamental class of $T^{(\{j\})}$. For a vertex $i \in \mathrm{Vert}(S)$ let $(\lambda_{i,1},\ldots,\lambda_{i,n})$ denote the coordinates of $\omega_i \in \Lambda_1$ in this basis.

LEMMA 3.3. *Let $I \in S_Q$, $I \neq \hat{0}$ be a simplex with the vertices $\{i_1, \dots, i_{n-q}\}$ listed in a positive order. Let $A = \{j_1 < \dots < j_q\} \subset [n]$ be a subset of indices, and let $e_A = e_{j_1} \wedge \dots \wedge e_{j_q} \in H_q(T^n;\mathbb{Z})$ be the fundamental class of $T^{(A)}$. Consider the map $\varrho\colon T^n \to T^n/T_I$. Then $\varrho_*(e_A) = C_{I,A}\Omega_I \in H_q(T^n/T_I;\mathbb{Z})$. The constant $C_{I,A}$ is equal to*

$$\operatorname{sgn}_A \det{(\lambda_{i,j})}_{\substack{i\in\{i_1,\dots,i_{n-q}\}\\ j\in[n]\setminus A}},$$

where $\operatorname{sgn}_A = \pm 1$ depends only on $A \subset [n]$. When $q = 0$, the constant $C_{I,A}$ is equal to ± 1 depending on the positivity of the basis $\omega_{i_1}, \dots, \omega_{i_n}$ and coincides with the sign of the fixed point of the action.

PROOF. When $q = 0$ the statement is simple. Let $q \neq 0$. Choose vectors $\gamma_1, \dots, \gamma_q$ so that $(b_l) = (\omega_{i_1}, \dots, \omega_{i_{n-q}}, \gamma_1, \dots, \gamma_q)$ is a positive basis of the lattice $H_1(T^n, \mathbb{Z})$. Thus $b_l = \mathcal{U}e_l$ with the matrix $\mathcal{U}$ of the form

$$\mathcal{U} = \begin{pmatrix} \lambda_{i_1,1} & \cdots & \lambda_{i_{n-q},1} & * & \cdots & * \\ \lambda_{i_1,2} & \cdots & \lambda_{i_{n-q},2} & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{i_1,n} & \cdots & \lambda_{i_{n-q},n} & * & \cdots & * \end{pmatrix}$$

We have $\det\mathcal{U} = 1$ since both bases are positive. Consider the inverse matrix $\mathcal{V} = \mathcal{U}^{-1}$. Thus

$$e_A = e_{j_1} \wedge \dots \wedge e_{j_q} = \sum_{M=\{\alpha_1<\dots<\alpha_q\}\subset[n]} \det{(\mathcal{V}_{j,\alpha})}_{\substack{j\in A\\ \alpha\in M}} b_{\alpha_1} \wedge \dots \wedge b_{\alpha_q}.$$

After taking the quotient $\Lambda/\langle\omega_{i_1}, \dots, \omega_{i_{n-q}}\rangle$ all summands with $M \neq \{n-q+1, \dots, n\}$ vanish. When $M = \{n-q+1, \dots, n\}$, the element $b_{n-q-1} \wedge \dots \wedge b_n = \gamma_1 \wedge \dots \wedge \gamma_q$ goes to Ω_I. Thus

$$C_{I,A} = \det{(\mathcal{V}_{j,\alpha})}_{\substack{j\in A\\ \alpha\in\{n-q+1,\dots,n\}}}.$$

Now apply Jacobi's identity (see e.g. [**4**, Sect.4]):

$$\det{(\mathcal{V}_{j,\alpha})}_{\substack{j\in A\\ \alpha\in\{n-q+1,\dots,n\}}} = \frac{(-1)^{\operatorname{sgn}}}{\det\mathcal{U}} \det{(\mathcal{U}_{r,s})}_{\substack{r\in\{1,\dots,n-q\}\\ s\in[n]\setminus A}}$$

where $\operatorname{sgn} = \sum_{r=1}^{n-q} r + \sum_{s\in[n]\setminus A} s$. Since first $n-q$ columns of $\mathcal{U}$ are exactly the vectors $\lambda_{i,j}$, this observation completes the proof . ☐

3.3. Face ring and linear system of parameters. Recall the definition of a face ring of a simplicial poset S (see [**15**] or [**5**]). For $I_1, I_2 \in S$ let $I_1 \vee I_2$ denote the set of least upper bounds, and $I_1 \cap I_2 \in S$ be the intersection of simplices (the intersection is well-defined and unique in the case when $I_1 \vee I_2 \neq \varnothing$).

DEFINITION 3.4. *The face ring* $\Bbbk[S]$ *is the quotient ring of* $\Bbbk[v_I \mid I \in S]$, $\deg v_I = 2|I|$ *by the relations*

$$v_{I_1} \cdot v_{I_2} = v_{I_1 \cap I_2} \cdot \sum_{J \in I_1 \vee I_2} v_J, \qquad v_\varnothing = 1.$$

The sum over an empty set is assumed to be 0.

Let $[m] = \{1, \ldots, m\}$ be the set of vertices of S and let $\Bbbk[m] = \Bbbk[v_1, \ldots, v_m]$ be the graded polynomial ring with $\deg v_i = 2$. The ring homomorphism $\Bbbk[m] \to \Bbbk[S]$ sending v_i to v_i defines a structure of $\Bbbk[m]$-module on $\Bbbk[S]$.

A characteristic function on Q determines the set of linear forms $\{\theta_1, \ldots, \theta_n\} \subset \Bbbk[S_Q]$, where $\theta_j = \sum_{i \in \mathrm{Vert}(S_Q)} \lambda_{i,j} v_i$. If $J \in S$ is a maximal simplex, $|J| = n$, then

$$\text{the matrix } (\lambda_{i,j})_{\substack{i \leqslant J \\ j \in [n]}} \text{ is invertible over } \Bbbk \tag{3.1}$$

by the $(*)$-condition. This condition is equivalent to the statement that the sequence $\{\theta_1, \ldots, \theta_n\}$ is a linear system of parameters in $\Bbbk[S]$ (see e.g.[**6**, Lm.3.5.8]). It generates an ideal $(\theta_1, \ldots, \theta_n) \subset \Bbbk[S]$ denoted by Θ.

A face ring $\Bbbk[S]$ is an algebra with straightening law (see, e.g. [**6**, §.3.5]). As a $\Bbbk$-module, it has an additive basis

$$\{P_\sigma = v_{I_1} \cdot v_{I_2} \cdot \ldots \cdot v_{I_t} \mid \sigma = (I_1 \leqslant I_2 \leqslant \ldots \leqslant I_t \in S)\}.$$

LEMMA 3.5. *The elements* $[v_I] = v_I + \Theta$ *span the* $\Bbbk$*-module* $\Bbbk[S]/\Theta$.

PROOF. Take any element P_σ with $|\sigma| \geqslant 2$. Using relations in the face ring, we can express $P_\sigma = v_{I_1} \cdot \ldots \cdot v_{I_t}$ as $v_i \cdot v_{I_1 \setminus i} \cdot \ldots \cdot v_{I_t}$ for a vertex $i \leqslant I_1$. Indeed, for every $J \in i \vee (I_1 \setminus i)$ except I_1, the product $v_J \cdot v_{I_2}$ vanishes.

The element v_i can be expressed as $\sum_{i' \nleqslant I_t} a_{i'} v_{i'}$ modulo Θ according to (3.1) (exclude all v_i corresponding to the vertices of some maximal simplex $J \geqslant I_t$). Thus $v_i v_{I_t}$ is expressed as a combination of $v_{I'_t}$ with $I'_t \overset{1}{>} I_t$. Therefore, up to ideal Θ, the element P_σ is expressed as a linear combination of elements $P_{\sigma'}$ which have either smaller length t (in case $|I_1| = 1$) or smaller I_1 (in case $|I_1| > 1$). By iterating this descending process, we express the element $P_\sigma + \Theta \in \Bbbk[S]/\Theta$ as a linear combination of $[v_I]$. □

4. Linear relations on face classes

Let $H^*_T(X)$ be a T^n-equivariant cohomology ring of X. Any proper face of Q is acyclic, therefore any face has a vertex. Hence, there is an injective homomorphism

$$\Bbbk[S_Q] \hookrightarrow H^*_T(X),$$

which sends v_I to the cohomology class, equivariant Poincare dual to $[X_I]$ (see [**9**, Lm.6.4]). The inclusion of a fiber in the Borel construction, $X \to X \times_T ET^n$, induces

the ring homomorphism $H_T^*(X) \to H^*(X)$. The subspace $H_*(X)$, Poincare dual to the image of the homomorphism

$$g\colon \Bbbk[S_Q] \hookrightarrow H_T^*(X) \to H^*(X) \tag{4.1}$$

is generated by the elements $[X_I]$, thus coincides with the submodule $\bigoplus_q (\dot{E}_X)_{q,q}^{\infty} \subset H_*(X)$. We call the classes $[X_I]$ the *face classes (or face cycles)* of X.

Note that the elements $[X_I] = [F_I] \otimes \Omega_I$ can also be considered as the free generators of the $\Bbbk$-module

$$\bigoplus_q (E_X)_{q,q}^1 = \bigoplus_q \bigoplus_{|I|=n-q} H_q(F_I, \partial F_I) \otimes H_q(T^n/T_I).$$

In the following let $\langle [X_I] \rangle$ denote the free $\Bbbk$-module generated by the elements $[X_I]$, $I \in S_Q$.

Proposition 4.1. *Let $C_{I,A}$ be the constants defined in Lemma 3.3. The submodule of $H_*(X)$ generated by the face classes $[X_I]$ has relations of the following two types:*

(1) *For each $J \in S$, $|J| = n-q-1$, and $A \subset [n]$, $|A| = q$ there is a relation $R_{J,A} = 0$ where*

$$R_{J,A} = \sum_{I, I \overset{1}{>} J} [I:J] C_{I,A} [X_I].$$

(2) *Let $q \leqslant n-2$ and let $\beta \in H_q(\partial Q)$ be a homology class lying in the image of the connecting homomorphism $\delta_{q+1}\colon H_{q+1}(Q, \partial Q) \to H_q(\partial Q)$. Let $\sum_{I, |I|=n-q} B_I [F_I]$ be a cellular chain representing β (such representation exists since every face of ∂Q is acyclic, thus may be considered as a homological cell). Then, for each $A \subset [n]$, $|A| = q$ we have a relation $R'_{\beta,A} = 0$, where*

$$R'_{\beta,A} = \sum_{I, |I|=n-q} B_I C_{I,A} [X_I].$$

Proof. The proof follows from the structure of homological spectral sequences of X and Y. The module $\bigoplus_q (E_X)_{q,q}^1$ is freely generated by $[X_I]$. Relations on $[X_I]$ in $H_*(X)$ appear as the images of the differentials hitting $\bigoplus_q (E_X)_{q,q}^1$. The relation of first type $R_{J,A}$ is the image of the generator

$$[F_J] \otimes [T^{(A)}] \in H_{q+1}(F_J, \partial F_J) \otimes H_q(T^n/T_J) \subset (E_X)_{q+1,q}^1$$

under the differential $(d_X)^1\colon (E_X)_{q+1,q}^1 \to (E_X)_{q,q}^1$. Thus relations of the first type span the image of the first differentials hitting $(E_X)_{q,q}^1$.

Let us prove that images of higher differentials are generated by $R'_{\beta,A}$. Higher differentials $(d_Q)^{\geqslant 2}$ coincide with $\delta_{*+1}\colon H_{*+1}(Q, \partial Q) \to H_*(\partial Q)$ by Proposition 2.5. The differentials $(d_Y)^{\geqslant 2}$ coincide with $\delta_{*+1} \otimes \mathrm{id}_\Lambda$ by Proposition 2.6. Thus the image

of $(d_Y)^{\geqslant 2}$ in $(E_Y)^2_{q,q}$ is generated by the elements $\beta\otimes[T^{(A)}]$, which are, in turn, the homology classes of the elements

$$\left(\sum_{I,|I|=n-q} B_I[F_I]\right)\otimes[T^{(A)}]\in(E_Y)^1_{q,q}.$$

By Proposition 2.7, the differential $(d_X)^*$ which hits $(E_X)^*_{q,q}$, coincides with the composition of $(d_Y)^*$ and inclusion f^2_*. The map $f^1_*\colon(E_Y)^1_{q,q}\to(E_X)^1_{q,q}$ is the sum of the maps

$$\mathrm{id}\otimes\varrho\colon H_q(F_I,\partial F_I)\otimes H_q(T^n)\to H_q(F_I,\partial F_I)\otimes H_q(T^n/T_I)$$

over all simplices I of rank $n-q$. Thus

$$f^2_*(\beta\otimes[T^{(A)}])=\left[f^1_*\left(\left(\sum_{I,|I|=n-q} B_I[F_I]\right)\otimes[T^{(A)}]\right)\right]=R'_{\beta,A}$$

by Lemma 3.3. □

REMARK 4.2. It follows from the spectral sequence that the element $R'_{\beta,A}\in\bigoplus_q(E_X)^2_{q,q}$ does not depend on a cellular chain, representing β. Proposition 2.7 also implies that relations $\{R'_{\beta,A}\}$ are linearly independent in $(E_X)^2_{q,q}$ when β runs over some basis of $\operatorname{Im}\delta_{q+1}$ and A runs over all subsets of $[n]$ of cardinality q.

Next we want to check that relations of the first type are exactly the relations in the quotient ring $\Bbbk[S_Q]/\Theta$.

PROPOSITION 4.3. *Let $\varphi\colon\langle[X_I]\rangle\to\Bbbk[S_Q]$ be the degree reversing linear map, which sends the generator $[X_I]$ to v_I. Then φ descends to the isomorphism of $\Bbbk$-modules*

$$\tilde\varphi\colon\langle[X_I]\rangle/\langle R_{J,A}\rangle\to\Bbbk[S_Q]/\Theta.$$

PROOF. (1) First we prove that $\tilde\varphi$ is well defined by showing that the element

$$\varphi(R_{J,A})=\textstyle\sum_{I,I\overset{1}{>}J}[I:J]C_{I,A}v_I\in\Bbbk[S_Q]$$

lies in Θ. Let $s=|J|$ and, consequently, $|I|=s+1$, $|A|=n-s-1$. Let $[n]\backslash A=\{\alpha_1<\ldots<\alpha_{s+1}\}$ and let $\{j_1,\ldots,j_s\}$ be the vertices of J listed in a positive order. Consider the $s\times(s+1)$ matrix:

$$\mathcal{D}=\begin{pmatrix}\lambda_{j_1,\alpha_1}&\ldots&\lambda_{j_1,\alpha_{s+1}}\\ \vdots&\ddots&\vdots\\ \lambda_{j_s,\alpha_1}&\ldots&\lambda_{j_s,\alpha_{s+1}}\end{pmatrix}$$

Denote by $\mathcal{D}_l$ the square submatrix obtained from $\mathcal{D}$ by deleting l-th column and let $a_l=(-1)^{l+1}\det\mathcal{D}_l$. We claim that

$$\varphi(R_{J,A})=\pm v_J\cdot(a_1\theta_{\alpha_1}+\ldots+a_{s+1}\theta_{\alpha_{s+1}}).$$

Indeed, after expanding each θ_l as $\sum_{i\in\mathrm{Vert}(S)}\lambda_{i,l}v_i$, all elements of the form v_Jv_i with $i<J$ cancel (the coefficients at these terms are determinants of matrices with two coinciding rows). Other terms give

$$\sum_{I,I\overset{1}{>}J,i=I\setminus J}(a_1\lambda_{i,\alpha_1}+\ldots+a_{s+1}\lambda_{i,\alpha_{s+1}})v_I.$$

The coefficient at v_I is equal to the determinant of the matrix

$$\begin{pmatrix}\lambda_{i,\alpha_1} & \ldots & \lambda_{i,\alpha_{s+1}}\\ \lambda_{j_1,\alpha_1} & \ldots & \lambda_{j_1,\alpha_{s+1}}\\ \vdots & \ddots & \vdots\\ \lambda_{j_s,\alpha_1} & \ldots & \lambda_{j_s,\alpha_{s+1}}\end{pmatrix}\tag{4.2}$$

by the cofactor expansion along the first row. This determinant is equal to $\mathrm{sgn}_A\,[I:J]C_{I,A}$ by the definition of $C_{I,A}$. Indeed, the number $C_{I,A}$ was defined as the determinant for some positive ordering of vertices of I. The ordering $i<j_1<\ldots<j_s$ (used to order the rows of matrix (4.2)) is either positive or negative depending on the incidence sign $[I:J]$.

(2) $\tilde{\varphi}$ is surjective by Lemma 3.5.

(3) Ranks of both spaces are equal. Indeed, $\dim\langle[X_I]\mid |I|=n-q\rangle/\langle R_{J,A}\rangle=\dim(E_X)^2_{q,q}=h'_{n-q}(S_Q)$ by Proposition 2.7. By Proposition 2.4 the poset S_Q is Buchsbaum. Thus $\dim(\Bbbk[S_Q]/\Theta)_{n-q}=h'_{n-q}(S_Q)$ by Schenzel's theorem (see [**14**], [**16**, Ch.II,§8.2], or [**11**, Prop.6.3] for simplicial posets).

(4) If $\Bbbk$ is a field, then we are done. Since the statement holds over any field, the case $\Bbbk=\mathbb{Z}$ automatically follows. $\square$

The Poincare duality in X yields

Corollary 4.4. *The map $g\colon\Bbbk[S_Q]\to H^*(X)$ factors through $\Bbbk[S_Q]/\Theta$ and the kernel of the homomorphism $\tilde{g}\colon\Bbbk[S_Q]/\Theta\to H^*(X)$ is additively generated by the elements*

$$L'_{\beta,A}=\sum_{I,|I|=n-q}B_IC_{I,A}v_I,$$

where $q\leqslant n-2$, $\beta\in\mathrm{Im}(\delta_{q+1}\colon H_{q+1}(Q,\partial Q)\to H_q(\partial Q))$, $\sum_{I,|I|=n-q}B_I[F_I]$ is a cellular chain in ∂Q representing β, and $A\subset[n]$, $|A|=q$.

Remark 4.5. The ideal $\Theta\subset\Bbbk[S_Q]$ coincides with the image of the homomorphism $H^{>0}(BT^n)\to H^*_T(X)$. So the fact that Θ vanishes in $H^*(X)$ is not surprising. An interesting thing is that Θ vanishes already in a second term of the spectral sequence, while other relations in $H^*(X)$ demonstrate the effects of higher differentials.

Note, that the elements $R'_{\beta,A} = \sum_{I,|I|=n-q} B_I C_{I,A}[X_I] \in (E_X)^2_{q,q}$ and $L'_{\beta,A} = \sum_{I,|I|=n-q} B_I C_{I,A} v_I \in \Bbbk[S_Q]/\Theta$ can be defined for any homology class $\beta \in H_q(\partial Q)$ (not only for the image of the connecting homomorphism δ_{q+1}).

Recall that the *socle* of a module $\mathcal{M}$ over the polynomial ring $\Bbbk[m]$ is the submodule

$$\operatorname{Soc}\mathcal{M} \stackrel{\text{def}}{=} \{y \in \mathcal{M} \mid \Bbbk[m]^+ \cdot y = 0\},$$

where $\Bbbk[m]^+$ is the maximal graded ideal of $\Bbbk[m]$.

THEOREM 2. *For every $\beta \in H_q(\partial Q)$, $q \leqslant n-1$ and $A \subset [n]$, $|A| = q$ the element $L'_{\beta,A} \in \Bbbk[S_Q]/\Theta$ lies in a socle of $\Bbbk[S_Q]/\Theta$.*

We postpone the proof to Section 8.

5. Non-face cycles of X

5.1. Spine and diaphragm cycles. In this section we give a geometrical description of homology classes of Q different from face classes.

CONSTRUCTION 5.1. Let $\eta \in H_k(Q)$ be a cycle of Q and let $a \in H_l(T^n)$, $l < k$ be a cycle of T^n represented by a subtorus $T^{(a)} \subset T^n$. They determine the homology class $\eta \otimes [T^{(a)}] \in H_{k,l}(Y) \cong H_k(Q) \otimes H_l(T^n)$. Thus they determine the class $\mathrm{Sp}_{\eta,a} \in H_{k+l}(X)$ via the isomorphism $f_*\colon H_{k,l}(Y) \to H_{k,l}(X)$ asserted by pt.(1) of Proposition 2.9. The cycles of this form (which are the elements of $H_{k,l}(X)$ for $k > l$) will be called *spine cycles (or spine classes)*.

Suppose that η is represented by an embedded pseudomanifold $N \subset Q$. We may assume that N lies in the interior of Q. Then the spine cycle $\mathrm{Sp}_{\eta,a}$ is represented by an embedded pseudomanifold $N \times T^{(a)} \subset Q^\circ \times T^n \subset X$.

CONSTRUCTION 5.2. Suppose $k < n$ and let $\zeta \in H_k(Q, \partial Q)$ be a relative homology class. Assume for simplicity that ζ is represented by a submanifold (or, generally, embedded pseudomanifold) $L \subset Q$ of dimension k with boundary $\partial L \subset \partial Q$ (which may be empty). Every proper face of Q is acyclic, thus can be considered as a homological cell of ∂Q. Therefore without lost of generality we may assume $\partial L \subset Q_{k-1}$. We also assume that $L\backslash\partial L \subset Q\backslash\partial Q$.

For each class $a \in H_l(T^n)$, represented by an l-dimensional subtorus $T^{(a)}$, consider the subset $Z_{L,a} = (L \times T^{(a)})/{\sim} \subset X$.

PROPOSITION 5.3.

- *If $l \geqslant k$, the subset $Z_{L,a}$ is a pseudomanifold. Thus it represents a well-defined element $\mathrm{Df}_{L,a} \in H_{k+l}(X)$ which will be called a diaphragm class (or diaphragm cycle).*
- *If $l > k$, the class $\mathrm{Df}_{L,a} \in H_{k+l}(X)$ depends only on the class $\zeta = [L] \in H_k(Q, \partial Q)$ but not on the particular representative L.*

PROOF. The set $((L\backslash\partial L)\times T^{(a)})/{\sim} = (L\backslash\partial L)\times T^{(a)}$ is a manifold of dimension $k+l$. The exceptional locus $(\partial L\times T^{(a)})/{\sim}$ has dimension at most $k+l-2$. Indeed, we have $\partial L\subset Q_{k-1}$, thus, under the projection map $(\partial L\times T^{(a)})/{\sim}\to\partial L$, every point $x\in\partial L$ has a preimage of the form $T^{(a)}/T_I$ with $|I|\geqslant n-k+1$. This set has dimension at most $l-1$ since $l+(n-k+1)>n$. Thus the total dimension of exceptional locus is at most $\dim\partial L+(l-1)=k+l-2$.

The second statement can be proved similarly. Let $(L_1,\partial L_1)$ and $(L_2,\partial L_2)$ be two manifolds representing the same element $\zeta\in H_k(Q,\partial Q)$. There exists a pseudomanifold bordism between them, i.e. a pseudomanifold Ξ of dimension $k+1$ with boundary and a map $\phi\colon\Xi\to Q$ such that L_1, L_2 are disjoint submanifolds of $\partial\Xi$, the restriction of ϕ to L_ϵ is the inclusion of $L_\epsilon\hookrightarrow Q$, and $\phi(\partial\Xi\backslash(L_1^\circ\sqcup L_2^\circ))\subset\partial Q$ (this follows from the geometrical definition of homology, see [**13**, App. A.2]). Again, we may assume that $\phi(\partial\Xi\backslash(L_1^\circ\sqcup L_2^\circ))\subset Q_k$. Similar to the first statement, we can consider the space $(\Xi\times T^{(a)})/{\sim}$ of dimension $k+l+1$. This space is a pseudomanifold with boundary, and the boundary is exactly the difference $Z_{L_1,a}-Z_{L_2,a}$. Thus $\mathrm{Df}_{L_1,a}=\mathrm{Df}_{L_2,a}$ in $H_*(X)$. ☐

Thus for $k<l$ there is a well defined homology class $\mathrm{Df}_{\zeta,a}\overset{\text{def}}{=}\mathrm{Df}_{L,a}\in H_{k,l}(X)$ depending on $\zeta\in H_k(Q,\partial Q)$ and $a\in H_l(T^n)$. These classes span the homology modules $H_{k,l}(X)$ for $k<l$ and correspond to pt.(2) of Proposition 2.9.

When $k=l<n$ we call the classes $\mathrm{Df}_{L,a}$ *extremal diaphragm classes*. In this case the situation is a bit different: the classes $\mathrm{Df}_{L,a}$ depend not only on the homology class of L but on the representative L itself. Nevertheless, if L_1 and L_2 represent the same class in $H_k(Q,\partial Q)$, then the classes $\mathrm{Df}_{L_1,a},\mathrm{Df}_{L_2,a}\in H_{k,k}(X)$ coincide modulo face classes, as proved below. Our goal is to derive exact formulas, thus we restrict to the case, when $a\in H_l(T^n)$ is represented by a coordinate subtorus $T^{(A)}$ for $A\subset[n]$, $|A|=l$.

CONSTRUCTION 5.4. Let $\phi_\epsilon\colon(L_\epsilon,\partial L_\epsilon)\to(Q,\partial Q)$, $\epsilon=1,2$, be two pseudomanifolds representing the same element $\zeta\in H_k(Q,\partial Q)$, $k<n$. As in the proof of Proposition 5.3 consider a pseudomanifold bordism $(\Xi,\partial\Xi)$ between L_1 and L_2, and the map $\phi\colon\Xi\to Q$, which sends the boundary ∂L into the union of L_1, L_2 and Q_k. The skeletal stratification of Q induces a stratification on Ξ. The restriction of the map ϕ sends Ξ_{k-1} to Q_{k-1}. Let δ be the connecting homomorphism

$$\delta\colon H_{k+1}(\Xi,\partial\Xi)\to H_k(\partial\Xi,\Xi_{k-1})$$

in the long exact sequence of the triple $(\Xi,\partial\Xi,\Xi_{k-1})$. Consider the sequence of homomorphisms

$$H_{k+1}(\Xi,\partial\Xi) \xrightarrow{\delta} H_{n-1}(\partial\Xi,\Xi_{k-1}) \cong$$
$$H_k(L_1,\partial L_1)\oplus H_k(L_2,\partial L_2)\oplus H_k(\partial\Xi\backslash(L_1^\circ\cup L_2^\circ),\partial\Xi_{k-1}) \xrightarrow{\mathrm{id}\oplus\mathrm{id}\oplus\phi_*}$$
$$H_k(L_1,\partial L_1)\oplus H_k(L_2,\partial L_2)\oplus H_k(Q_k,Q_{k-1}).$$

It sends the fundamental cycle $[\Xi]\in H_{k+1}(\Xi,\partial\Xi)$ to the element

$$\left([L_1],-[L_2],\sum_{I,\dim F_I=k} D_I[F_I]\right) \tag{5.1}$$

of the group $H_k(L_1,\partial L_1)\oplus H_k(L_2,\partial L_2)\oplus H_k(Q_k,Q_{k-1})$, for some numbers $D_I\in\Bbbk$.

Proposition 5.5. *Let L_1,L_2 be two manifolds representing the same class $\zeta\in H_k(Q,\partial Q)$, $k<n$. Consider any subset $A\subset[n]$, $|A|=k$ and let $a\in H_k(T^n)$ be the fundamental class of the coordinate subtorus $T^{(A)}$. Then there is a relation in $H_{2k}(X)$:*

$$\mathrm{Df}_{L_1,a}-\mathrm{Df}_{L_2,a}+\sum_{I,\dim F_I=k} D_IC_{I,A}[X_I]=0. \tag{5.2}$$

The numbers D_I are given by (5.1)*, and the numbers $C_{I,A}$ were defined in Lemma 3.3.*

Proof. Consider the space $(\Xi\times T^{(A)})/\sim'$ and the map $\phi\times\iota\colon(\Xi\times T^{(A)})/\sim'\to X=(Q\times T^n)/\sim$, where the relation $\sim'$ is induced from $\sim$ by the map ϕ, and $\iota\colon T^{(A)}\to T^n$ is the inclusion map. The space $(\Xi\times T^{(A)})/\sim'$ is a pseudomanifold with boundary, and its boundary represents the element $\mathrm{Df}_{L_1,A}-\mathrm{Df}_{L_2,A}+\sum_{I,\dim F_I=k}D_IC_{I,A}[X_I]$ in $H_{2k}(X)$ by Lemma 3.3. Thus this element vanishes in homology. $\square$

Therefore, up to face classes, the middle homology group $H_{k,k}(X)$ coincides with $H_k(Q,\partial Q)\otimes H_k(T^n)$ for $k<n$. This was stated in equivalent form in pt.(3) of Proposition 2.9.

5.2. Integral coefficients. Proposition 2.9 was stated only over a field. On the other hand, the geometrical constructions of the previous subsection determine the additive homomorphisms

$$\bigoplus_{k>l} H_k(Q)\otimes H_l(T^n)\to H_*(X),\qquad \bigoplus_{k\leqslant l} H_k(Q,\partial Q)\otimes H_l(T^n)\to H_*(X)$$

over $\mathbb{Z}$ as well. Combining this with the inclusion of face cycles

$$\langle[X_I]\rangle/(R_{J,A},R'_{\beta,A})\hookrightarrow H_*(X)$$

we get a map

$$\left(\bigoplus_{k>l} H_k(Q)\otimes H_l(T^l)\right)\oplus\left(\bigoplus_{k\leqslant l} H_k(Q,\partial Q)\right)\oplus\left(\langle[X_I]\rangle/(R_{J,A},R'_{\beta,A})\right)\to H_*(X) \tag{5.3}$$

which is well defined for any coefficients and is an isomorphism over a field. Now it follows by induction from the universal coefficient theorem, that the map (5.3) is an isomorphism over $\mathbb{Z}$. This proves

PROPOSITION 5.6. *Proposition 2.9 holds over $\mathbb{Z}$. Homology groups of X are generated by face classes, spine classes, and diaphragm classes. The groups $H_{k,l}(X)$ for $k > l$ are generated by spine classes; the groups $H_{k,l}(X)$ for $k < l$ are generated by non-extremal diaphragm classes; the short exact sequence*

$$0 \to (\dot{E}_X)_{k,k}^{\infty} \to H_{k,k}(X) \to H_k(Q, \partial Q) \otimes \Lambda_k \to 0.$$

identifies the quotient of $H_{k,k}(X)$ by the face classes with the group of extremal diaphragm classes. This short exact sequence splits, but the splitting is not canonical.

5.3. Auxiliary cycles and relations. In construction 5.1 we defined the cycles $\mathrm{Sp}_{\eta,a}$ for each $\eta \in H_k(Q)$ and $a \in H_l(T^n)$ under assumption that $k > l$. But the same construction can be applied for any k and l. If η is represented by a pseudomanifold N in the interior of Q and a is represented by a subtorus, then the product $N \times T^{(a)}$ is a submanifold in $Q^{\circ} \times T^n \subset X$, thus represents an element $[N \times T^{(a)}] \in H_{k+l}(X)$.

Although for $k \leqslant l$ the element $N \times T^{(a)}$ is not a spine cycle, we keep denoting it $\mathrm{Sp}_{\eta,a}$. For $k < l$ we have $\mathrm{Sp}_{\eta,a} = \mathrm{Df}_{\eta',a}$, where η' is the image of η in $H_k(Q, \partial Q)$.

6. Intersections in $H_*(X)$

Let $\cap\colon H_k(M) \otimes H_l(M) \to H_{k+l-\dim M}(M)$ denote the intersection product on a closed manifold M, i.e. an operation Poincare dual to the cup-product in cohomology. From the geometrical structure of X (and also from Proposition 4.3) follows

PROPOSITION 6.1. *If $I_1, I_2 \in S_Q$, then $[X_{I_1}] \cap [X_{I_2}] = [X_{I_1 \cap I_2}] \cap \sum_{J \in I_1 \vee I_2} [X_J]$.*

Intersections of spine and diaphragm classes can also be described geometrically. There exists an intersection product, $\cap\colon H_{k_1}(Q) \otimes H_{k_2}(Q, \partial Q) \to H_{k_1+k_2-n}(Q)$ dual to the cohomology product $H^{n-k_1}(Q, \partial Q) \otimes H^{n-k_2}(Q) \to H^{2n-k_1-k_2}(Q, \partial Q)$. If the classes $\eta \in H_{k_1}(Q)$ and $\zeta \in H_{k_2}(Q, \partial Q)$ are represented by a smooth submanifold N and a smooth submanifold with boundary $(L, \partial L)$ respectively, and if N intersects L transversely, then $\eta \cap \zeta$ is represented by a submanifold $N \cap L$. There is also an intersection product in homology of torus $\cap\colon H_{l_1}(T^n) \otimes H_{l_2}(T^n) \to H_{l_1+l_2-n}(T^n)$. From the geometrical construction of cycles in X we conclude that the intersection product on $H_*(X)$ has the following structure.

PROPOSITION 6.2.

(1) *The cycles $\mathrm{Sp}_{\eta,a} \in H_{k_1,l_1}(X)$, $k_1 > l_1$ and $\mathrm{Df}_{L,b} \in H_{k_2,l_2}(X)$, $k_2 \leqslant l_2$ satisfy*

$$\mathrm{Sp}_{\eta,a} \cap \mathrm{Df}_{L,b} = \mathrm{Sp}_{\eta \cap [L], a \cap b}\,.$$

Since $\dim(\eta \cap [L]) = k_1 + k_2 - n$ *and* $\dim(a \cap b) = l_1 + l_2 - n$*, the element* $\mathrm{Sp}_{\eta\cap[L],a\cap b}$ *is either a spine class (if* $k_1 + k_2 > l_1 + l_2$*), or a diaphragm class determined in subsection 5.3 (if* $k_1 + k_2 \leqslant l_1 + l_2$*).*

(2) *The cycles* $\mathrm{Sp}_{\eta',a} \in H_{k_1,l_1}(X)$*,* $k_1 > l_1$ *and* $\mathrm{Sp}_{\eta'',b} \in H_{k_2,l_2}(X)$*,* $k_2 > l_2$ *satisfy*

$$\mathrm{Sp}_{\eta',a} \cap \mathrm{Sp}_{\eta'',b} = \mathrm{Sp}_{\eta'\cap\eta'',a\cap b}.$$

The result is a spine cycle.

(3) *Spine cycles do not meet face cycles:* $\mathrm{Sp}_{\eta,a} \cap [X_I] = 0$*.*

The proof follows directly from the constructions.

PROPOSITION 6.3. *The linear span of proper face classes* $[X_I]$ *is an ideal of* $H_*(X)$ *with respect to the intersection product.*

PROOF. Suppose $I \neq \hat{0}$ and let $\imath\colon X_I \hookrightarrow X$ be the inclusion of a face submanifold. Let κ be a cohomology class Poincare dual to some diaphragm class Df in X. Then $[X_I] \cap \mathrm{Df} = [X_I] \frown \kappa = \imath_*(\imath^*(\kappa) \frown [X_I])$. The class $\imath^*(\kappa) \frown [X_I] \in H_*(X_I)$ is a linear combination of face classes since there are no other classes in $H_*(X_I)$. Thus $[X_I] \cap \mathrm{Df}$ is a linear combination of face classes in $H_*(X)$. $\square$

REMARK 6.4. The description of intersections of diaphragm cycles with themselves and with face cycles is difficult in general. Nevertheless, in practice one can use the following trick (cf. the discussion of a similar problem in [**3**, Sect.8]). Suppose the task is to compute $\mathrm{Df}_{L,a} \cap [X_I]$. If $L \cap F_I = \varnothing$, then the intersection product is 0. If not, find another submanifold with boundary L' such that $[L'] = [L] \in H_*(Q, \partial Q)$ and $L' \cap F_I = \varnothing$. Then, by Proposition 5.5 we have $\mathrm{Df}_{L,a} = \mathrm{Df}_{L',a} + \Sigma$, where Σ is a linear combination of face classes. Thus we have

$$\mathrm{Df}_{L,a} \cap [X_I] = \mathrm{Df}_{L',a} \cap [X_I] + \Sigma \cap [X_I] = \Sigma \cap [X_I].$$

The last intersection can be computed by Proposition 6.1.

7. Examples

7.1. A very concrete example. This example is similar to the one studied in [**12**, Th.3.1]. For Q take a square with triangular hole. Orientations of facets and values of characteristic function are assigned to Q as shown on Fig.1, left.

Homology groups of the corresponding 4-dimensional manifold $X = (Q \times T^2)/\sim$ are as follows.

(1) Face classes. These are the following: the fundamental class $[X] \in H_4(X)$; the classes of characteristic submanifolds

$$[X_1], [X_2], \ldots, [X_7] \in H_{1,1}(X) \subset H_2(X),$$

which correspond to the sides of Q; and the classes of fixed points

$$[X_{12}], [X_{23}], [X_{34}], [X_{14}], [X_{56}], [X_{67}], [X_{57}] \in H_{0,0}(X) = H_0(X),$$

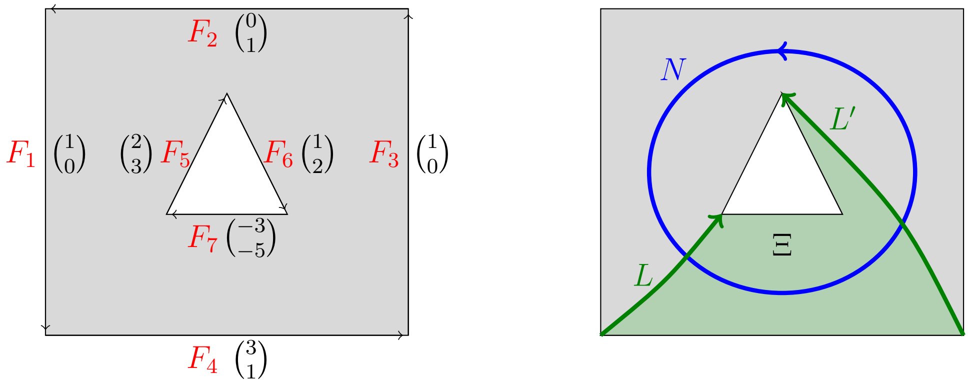

FIGURE 1. Structure of Q and values of characteristic function

which correspond to vertices of Q. Relations on these classes are given by Proposition 4.1. The first-type relations in $H_2(X)$ are:

$$[X_1] + [X_3] + 3[X_4] + 2[X_5] + [X_6] - 3[X_7] = 0$$

$$[X_2] + [X_4] + 3[X_5] + 2[X_6] - 5[X_7] = 0$$

(the coefficients are respectively first and second coordinates of the values of characteristic function). The first type relations on the classes $[X_{ij}]$ are encoded by the sides of of the orbit space. These relations are the following:

$$[X_{12}] = -[X_{23}] = [X_{34}] = -[X_{14}];$$

$$[X_{56}] = [X_{67}] = [X_{57}].$$

(the signs are due to the signs of fixed points). To find relations of the second type, we need to pick a homology class in the image of $\delta_1\colon H_1(Q, \partial Q) \to H_0(\partial Q)$. Take for example the class, represented by the chain $[F_{14}] - [F_{57}]$. It gives a relation of the second type

$$[X_{14}] - [X_{57}] = 0.$$

Thus all fixed points up to sign represent the same generator $[\mathrm{pt}] \in H_0(X)$. Of course, this follows easily from the connectivity of X, but here we wanted to emphasize the different nature of two types of relations.

(2) Spine cycles. Consider a submanifold $N \subset Q$ representing the generator $\eta \in H_1(Q)$ (Fig.1, right). Together with the class of a point $[T^{(\varnothing)}] \in H_0(T^2)$ it determines a spine class $\mathrm{Sp}_{\eta,\varnothing} \in H_{1,0}(X) = H_1(X)$. Geometrically, $\mathrm{Sp}_{\eta,\varnothing}$ is represented by a submanifold $N \subset Q$ lifted by a zero-section map $Q \hookrightarrow X$.

(3) Diaphragm cycles. Consider the submanifold L representing the generator of $H_1(Q, \partial Q)$ with the boundary lying in the 0-skeleton of Q (see Fig.1, right). For each subset $A = \{1\}, \{2\}, \{1, 2\}$ we have a homology cycle in $H_{1,|A|}(X)$ represented

by a pseudomanifold $(L \times T^{(A)})/\sim$. Thus we have the generators

$$\mathrm{Df}_{L,1} = [(L \times T^{(\{1\})})/\sim], \qquad \mathrm{Df}_{L,2} = [(L \times T^{(\{2\})})/\sim]$$

of $H_{1,1}(X) \subset H_2(X)$ and the generator

$$\mathrm{Df}_{L,\{12\}} = [(L \times T^2)/\sim]$$

of $H_{1,2}(X) = H_3(X)$. Let L' be another submanifold representing the same homology class in $H_1(Q, \partial Q)$ (see Figure 1). Consider a bordism Ξ between L and L' shown on the figure. We have

$$\mathrm{Df}_{L,\{12\}} = \mathrm{Df}_{L',\{12\}}$$

in $H_3(X)$ since $(\Xi \times T^2)/\sim$ is a pseudomanifold bordism between $(L \times T^2)/\sim$ and $(L' \times T^2)/\sim$.

We have a relation $\delta\Xi = -[L_1] + [L_2] + [F_4] + [F_6] + [F_7]$. It generates the relations

$$-\mathrm{Df}_{L,1} + \mathrm{Df}_{L',1} + 1[X_4] + 2[X_6] - 5[X_7] = 0$$
$$-\mathrm{Df}_{L,2} + \mathrm{Df}_{L',2} + 3[X_4] + 1[X_6] - 3[X_7] = 0$$

in $H_2(X)$. These relations are the boundaries of $(\Xi \times T^{(\{1\})})/\sim$ and $(\Xi \times T^{(\{2\})})/\sim$ respectively. The coefficients are the complimentary coordinates of characteristic function: for the cycle encoded by the first coordinate subtorus, we take the second coordinates of characteristic function, and vice versa. In this computation we used the formula for the coefficients $C_{I,A}$ asserted by Lemma 3.3.

(4) Intersections of cycles can be seen from the picture. In particular, the cycles $\mathrm{Sp}_{N,\varnothing}$ and $\mathrm{Df}_{L,\{1,2\}}$ are transversal, and their intersection induces a nondegenerate pairing between $H_1(X)$ and $H_3(X)$.

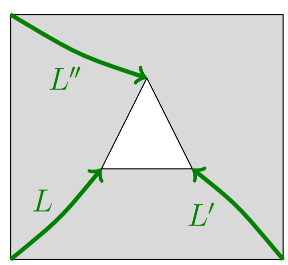

FIGURE 2. Different representatives of diaphragm classes

For a nontrivial example, let us compute the intersection of $\mathrm{Df}_{L,1}$ with $\mathrm{Df}_{L,2}$ to demonstrate the idea sketched in Remark 6.4. Consider the auxiliary intervals L' and L'' shown on Fig.2. Similar to the calculations above we have

$$\mathrm{Df}_{L,1} = \mathrm{Df}_{L',1} + 1[X_4] - 5[X_7]$$

$$\mathrm{Df}_{L,2} = \mathrm{Df}_{L'',2} - 1[X_1] - 2[X_5]$$

Thus $\mathrm{Df}_{L,1}\cap\mathrm{Df}_{L,2} = (\mathrm{Df}_{L',1}+[X_4]-5[X_7])\cap(\mathrm{Df}_{L'',2}-[X_1]-2[X_5]) = -[X_4]\cap[X_1]+10[X_7]\cap[X_5] = -[X_{14}]+10[X_{57}] = 9[pt]\in H_0(X)$.

7.2. Toric origami manifolds. In this subsection we apply the general method to a class of toric origami manifolds and derive some results proved in [**3**] in a different way.

Toric origami manifolds (see [**7**],[**10**]) appeared in differential geometry as generalizations of symplectic toric manifolds. The precise geometrical definition is in most part irrelevant to our study. Essential are the following properties: orientable toric origami manifold X is a manifold with locally standard torus action; its orbit space $Q = X/T^n$ is homotopy equivalent to a graph Γ, and inclusion of any face in Q is homotopy equivalent to the inclusion of a subgraph in Γ.

As usual, there is a principal torus bundle $Y \to Q$ such that $X = Y/{\sim}$. Since Q is homotopy equivalent to a graph, $H^2(Q,\mathbb{Z}^n) = 0$, so the Euler class of Y vanishes. Thus in origami case we have $Y = Q\times T^n$.

Now we restrict to the case when all proper faces of Q are acyclic. Since they are homotopy equivalent to graphs, it follows automatically that they are contractible. Let $b_1 = \dim H_1(Q) = \dim H_1(\Gamma)$. Poincare–Lefchetz duality implies:

$$H_q(Q,\partial Q)\cong H^{n-q}(Q)\cong\begin{cases}\mathbb{Z}, \text{ if } q=n;\\ \mathbb{Z}^{b_1}, \text{ if } q=n-1;\\ 0, \text{ otherwise.}\end{cases}$$

Let us describe the connecting homomorphisms $\delta_i\colon H_i(Q,\partial Q)\to H_{i-1}(\partial Q)$. For simplicity we discuss the case $n\geqslant 4$; dimensions 2 and 3 can be done similarly. When $n\geqslant 4$, lacunas in the exact sequence of the pair $(Q,\partial Q)$ imply that $\delta_i\colon H_i(Q,\partial Q)\to H_{i-1}(\partial Q)$ is an isomorphism for $i = n-1, n$, and trivial otherwise. We have

$$H_i(\partial Q)\cong\begin{cases}\Bbbk, \text{ if } i=0,n-1;\\ \Bbbk^{b_1}, \text{ if } i=1,n-2;\\ 0, \text{ otherwise.}\end{cases}$$

Proposition 2.7 implies that $(E_X)^2_{p,q}$ has the form shown schematically on Figure 3. The differential $(d_X)^2$ hitting the marked position produces relations $R'_{\beta,A}$ of the second type on the cycles $[X_I]\in H^{2n-4}(X)$. These relations are explicitly described by Proposition 4.1, and the number of independent relations is $\binom{n}{2}b_1$. Dually, this consideration shows that the map $\Bbbk[S]/\Theta\to H^*(X)$ has nontrivial kernel of dimension $\binom{n}{2}b_1$ in degree 4.

In addition, we have the following non-face cycles:

(1) There are b_1 one-dimensional spine classes, which appear as the liftings of cycles in Γ.

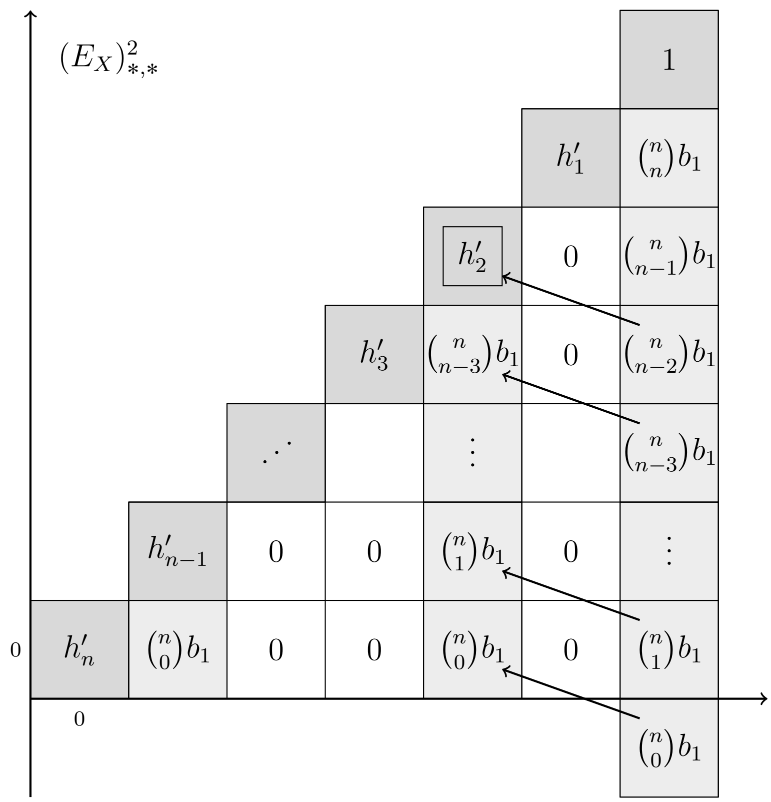

FIGURE 3. Spectral sequence for homology of orientable origami manifold with acyclic proper faces of the orbit space. Instead of entries we write their ranks.

(2) There are b_1 diaphragm classes of codimension 1 given by the generators of $H_{n-1}(Q,\partial Q) \cong H^1(\Gamma)$ swept by the action of a whole torus. These cycles are equivariant. They are dual to cycles (1).
(3) There are nb_1 extremal diaphragm classes of codimension 2. These are given by the generators of $H_{n-1}(Q,\partial Q)$ swept around by actions of $(n-1)$-dimensional subtori. A choice of these classes is not canonical.

8. Collar model

In this section we prove Theorem 2 using an auxiliary space $\widehat{X}$.

CONSTRUCTION 8.1. Consider the space $\widehat{Q} = \partial Q \times [0,1]$. It is an $(n-1)$-dimensional manifold with the boundary $\partial\widehat{Q}$ of the form $\partial_0\widehat{Q} \sqcup \partial_1\widehat{Q}$, where $\partial_\epsilon\widehat{Q} = \partial Q \times \{\epsilon\}$, $\epsilon = 0,1$. We may identify $\partial_0\widehat{Q}$ with ∂Q and consider $\widehat{Q}$ as a filtered topological space:

$$Q_0 \subset Q_1 \subset \ldots \subset Q_{n-1} = \partial Q = \partial_0\widehat{Q} \subset \widehat{Q}.$$

One may think about $\widehat{Q}$ as a collar of ∂Q inside Q.

Consider the space $\widehat{Y} = \widehat{Q} \times T^n$ and the identification space $\widehat{X} = \widehat{Y}/\sim$. The relation $\sim$ identifies points over $\partial_0\widehat{Q}$ in the same way as it does for $\partial Q \subset Q$, while there are no identifications over $\partial_1\widehat{Q}$. The space $\widehat{X}$ is a manifold with boundary. Its boundary consists of points over $\partial_1 Q$, hence is homeomorphic to $\partial Q \times T^n$. The space $\widehat{X}$ can be considered as a T^n-invariant tubular neighborhood of the union of all characteristic submanifolds in X. There are natural topological filtrations on $\widehat{Y}$ and $\widehat{X}$ induced by the filtration on $\widehat{Q}$.

In terminology of [**2**] the space $\widehat{X}$ is a Buchsbaum pseudo-cell complex, thus Propositions 2.5, 2.6, and items (1)–(5) of Proposition 2.7 hold for $\widehat{Q}$, $\widehat{Y}$, and $\widehat{X}$. The n-th column of all spectral sequences vanishes, since $H_*(\widehat{Q}, \partial_0\widehat{Q}) = 0$. Thus all spectral sequences $(\dot{E}_{\widehat{Q}})^r$, $(\dot{E}_{\widehat{Y}})^r$, $(\dot{E}_{\widehat{X}})^r$ collapse at a first page and, consequently, the spectral sequences $(E_{\widehat{Q}})^r$, $(E_{\widehat{Y}})^r$, $(E_{\widehat{X}})^r$ collapse at a second page.

For each $I \in S_Q$, with $\dim F_I = q < n$ there is a distinguished element $[X_I] \in H_{2q}(\widehat{X}_q, \widehat{X}_{q-1}) = (E_{\widehat{X}})^1_{q,q}$. It survives in a spectral sequence, and gives the fundamental class of the face submanifold $X_I \subset \widehat{X}$. Linear relations on classes $[X_I]$ in $H_*(\widehat{X})$ are described similarly to Section 4. When $q = n-1$ there are no relations on $[X_I]$, since there are no differentials landing at the cell $(E_{\widehat{X}})^1_{n-1,n-1}$. For $q < n-1$ the relations on $[X_I]$ are the images of $(d_{\widehat{X}})^1\colon (E_{\widehat{X}})^1_{q+1,q} \to (E_{\widehat{X}})^1_{q,q}$. These differentials coincide with $(d_X)^1$ thus the relations on $[X_I]$ for $|I| > 1$ are exactly $R_{J,A}$ defined in Proposition 4.1. Thus Proposition 4.3 implies

Lemma 8.2. *Let V_* be a submodule of $H_*(\widehat{X})$ generated by face classes $[X_I]$, $I \neq \hat{0}$. Then there is a degree reversing linear map*

$$\tilde{\varphi}\colon V_{2q} \to (\Bbbk[S_Q]/\Theta)_{2(n-q)}.$$

sending $[X_I]$ to v_I. It is an isomorphism for $q < n-1$ and surjective for $q = n-1$. This map is a ring homomorphism with respect to the intersection product on $\widehat{X}$.

Now we can give a geometrical proof of Theorem 2.

Proof. We need to prove that the element $L'_{\beta,A}$ lies in a socle of $\Bbbk[S_Q]/\Theta$. Thus we need to show that $L'_{\beta,A}v_i = 0$ for every vertex $i \in \mathrm{Vert}(S_Q)$, where $\beta \in H_q(\partial Q)$, $|A| = q$, and $q \leqslant n-2$. According to Lemma 8.2, it is sufficient to show that $R'_{\beta,A} \cap [F_i] = 0$ in $H_*(\widehat{X})$.

Consider a geometrical cycle $\mathcal{B} \subset \partial_0\widehat{Q}$ representing β. Now we allow $\mathcal{B}$ be any representative, and do not require that it lies in the stratum Q_q. Consider the cycle $\mathcal{B}\times T^{(A)}$ in $\partial_0\widehat{Q}\times T^n$, and the corresponding cycle $(\mathcal{B}\times T^{(A)})/\sim$ in $\widehat{X}$. The latter cycle represents the class $R'_{\beta,A} \in H_{2q}(\widehat{X})$ by definition. The space $\widehat{Q}$ is homologous to $\partial_0\widehat{Q}$

so we can move the cycle $\mathcal{B} \subset \widehat{Q}$ away from the boundary $\partial_0\widehat{Q}$. Thus $\mathcal{B} \cap [F_i] = \varnothing$ and therefore $R'_{\beta,A} \cap [X_i] = 0$ in $H_*(\widehat{X})$. □

REMARK 8.3. The same argument proves that $L'_{\beta,A}v_I = 0$ for any simplex $I \in S \backslash \hat{0}$. This fact does not directly follow from Theorem 2, since the map $\Bbbk[m] \to \Bbbk[S]/\Theta$ may be not surjective in general.

REMARK 8.4. The only reason why we considered the collar model $\widehat{X}$ instead of X is that there are no additional relations in $H_*(\widehat{X})$ compared with $\Bbbk[S_Q]/\Theta$. The space $\widehat{X}$ captures the properties of $\Bbbk[S_Q]/\Theta$ more precisely than X. On the other hand, $\widehat{X}$ is a manifold with boundary so there is a geometrical intersection theory on it. This makes it an object worth studying.

REMARK 8.5. The classes $R'_{\beta,A} \in (E_X)^2_{q,q}$ are the images of the classes $\beta \times [T^{(A)}] \in H_q(\partial Q) \times H_q(T^n)$ under the homomorphism $\dot{f}^1_*\colon (\dot{E}_{\widehat{Y}})^1_{q,q} \to (\dot{E}_{\widehat{X}})^1_{q,q}$. This homomorphism is injective by Proposition 2.7, thus the construction gives an inclusion

$$H_q(\partial Q) \otimes H_q(T^n) \hookrightarrow \mathrm{Soc}(\Bbbk[S_Q]/\Theta)_{2(n-q)}$$

for each $q \leqslant n-2$. When $q = n-1$, the map $H_{n-1}(\partial Q) \otimes H_{n-1}(T^n) \to \mathrm{Soc}(\Bbbk[S_Q]/\Theta)_2$ has the kernel of the form $\langle[\partial Q]\rangle \otimes H_{n-1}(T^n)$, where $[\partial Q]$ denotes the fundamental class of ∂Q. Note that ∂Q may be disconnected, thus there could exist classes in $H_{n-1}(\partial Q)$ different from $[\partial Q]$. So far, there exists an injective map

$$(H_{n-1}(\partial Q)/\langle[\partial Q]\rangle) \otimes H_{n-1}(T^n) \hookrightarrow \mathrm{Soc}(\Bbbk[S_Q]/\Theta)_2$$

These statements reprove the result of Novik–Swartz [**11**, Th.3.5] in case of homology manifolds.

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